Optimizing the Management of Uneven-aged Forest Stands

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Two perennial problems in the management of uneven-aged forests are considered: (i) determination of the optimal sustainable distribution of trees by diameter class, i.e. stand structure, for a given initial stocking level, and (ii) the optimal cutting schedule for the conversion of an irregular stand to a target structure. It is shown, using examples for northern hardwood stands in Wisconsin, that both problems can be solved via mathematical programming techniques. The programming approaches utilize a set of nonlinear equation models for stand table projections which consider the interdependence of size classes within the stand. To illustrate procedures, optimal stand structures are found for a case where initial stand basal area is constrained to specified levels and the objective is to maximize value growth over the cutting cycle. A conversion cutting schedule is then determined for a case in which the objective is maximization of present worth. It is emphasized that both the optimal distribution and conversion problems can be generalized to consider a broad range of objective functions, lengths of cutting cycle, and constraints on the growing stock.


L’article traite de deux vieux problèmes dans l’aménagement des forêts inéquilières: (i) la détermination de la distribution optimale des arbres par classe de diamètre, i.e. la structure du peuplement pour un stock initial donné, et (ii) la séquence optimale de coupe pour la conversion d’un peuplement irrégulier en vue d’une structure projetée. A l’aide d’exemples pour des peuplements de feuillus nordiques au Wisconsin, on montre que ces deux problèmes peuvent se résoudre par les techniques de programmation mathématique. Les approches de programmation utilisent un ensemble d’équations non-linéaires pour les projections de tables de peuplement qui tiennent compte de l’interdépendance des classes de diamètre à l’intérieur du peuplement. Pour illustrer ces approches, les structures optimales de peuplement sont trouvées pour un cas où la surface terrière initiale du peuplement est restreinte à des niveaux spécifiés et où l’objectif est de maximiser la valeur économique de l’accroissement durant la période de rotation. Un programme de coupe de conversion est alors déterminé pour un cas où l’objectif est la maximisation de la valeur présente. On montre comment les problèmes de distribution optimale et de conversion peuvent être généralisés pour considérer un large éventail de fonctions objectives, longueurs de rotation, et contraintes sur le stock en place.

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Introduction

A fundamental component of guidelines for the management of forest stands is the specification of stocking levels appropriate to various yield-oriented objectives. In uneven-aged stands, it is also important to identify appropriate

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basis for these approaches is the recent development of relatively simple growth models such as those described by Leary (1970), Moser (1973), and Ek (1974). These models explicitly consider the interdependence of stems within a stand and hence provide a means for predicting stand response to changes in structure due to harvesting. The intent of this paper is to describe approaches for determining optimal structure, stocking, and transition strategies for uneven-aged stands. Examples are presented using a model developed for northern hardwood stands in north central Wisconsin.

**The Growth Model**

The model of uneven-aged forest growth which forms the basis of the present study is a modification of a system developed by Ek (1974). The data utilized for developing the model were obtained from 1964 and 1969 measurements of tree diameters and heights on 132 permanent 1/7 acre (0.058 ha) plots located on Owens-Illinois, Inc. woodlands in central and northern Wisconsin. These stands were at least 50% sugar maple by basal area in 1964. Other stand components included yellow birch, white birch, northern red oak, trembling aspen, and balsam fir. All live trees greater than 4.96 in. (12.6 cm) d.b.h. o.b. were measured. For analysis, tree data were grouped by 2 in. (5.1 cm) diameter classes based on the 1964 tally (e.g. the 8 in. (20.3-cm) class boundaries were 7.0–8.9 in. (17.8–22.6 cm)). This produced 518 diameter class records.

This data represented most of the remeasured northern hardwood plots from a low-intensity systematic sample of company woodlands. Probably all of these plots were subject to some type of cutting in the past, but none used here were cut during the growth period. Several were thinned or partially cut in the decade prior to 1964. Diameter distributions of the plots indicated a mixture of negative exponential, unimodal, and bimodal shapes. Little age data was available, but it was evident that both even- and uneven-aged stands were represented in the data set.

The components of stand growth (ingrowth, mortality, and survivor growth) for a 5 year period are described by the following expressions:

**Ingrowth** (number of new trees growing into the smallest measured diameter class, *i.e.* the 6.0 in. (15.2 cm) class):

\[
I = 7.07933(Y_{21}X_{1} + \ldots + Y_{2N}X_{N})/ (Y_{1} + \ldots + X_{N})^{1.40072}
\]

\[r^2 = 0.72, \text{Standard error = 23.83}\]

Upgrowth (number of trees rising from diameter class *D* to class *D* + 1):

\[
U_{D} = 0.00330X_{D}^{0.88218} S Y_{1D}^{0.48383} \times \exp (-0.00286(Y_{21}X_{1} + \ldots + Y_{2N}X_{N}))
\]

\[r^2 = 0.76, \text{Standard error = 8.74}\]

Natural mortality in diameter class *D* (number of trees):

\[
M_{L} = 0.04109 X_{D} \text{ for } D = 1, \ldots, N,
\]

\[r^2 = 0.31, \text{Standard error = 4.06}\]

where all terms are on a per acre (0.405 ha) basis and

*X_{D}* is the number of trees in diameter class *D* at the beginning of the growth period, there being *N* initial diameter classes;

\[Y_{11}, \ldots, Y_{1N} \text{ are the class midpoint diameters at breast height, *i.e.* } 6.0, 8.0, \text{ etc.;}
\]

\[Y_{21}, \ldots, Y_{2N} \text{ are tree basal areas}^2 \text{ in square feet corresponding to class midpoint diameters for diameter classes } 1, 2, \ldots, N; \text{ and}
\]

*S* is a site measure analogous to site index, with *S* assuming values of 40, 50, 60, and 70. Further details on the data set and model derivation and fitting are given by Ek (1974).

Given the above growth component expressions, the number of trees in each diameter class at the *end* of a growth period (*t*) can be written as:

\[X_{t}(t) = X_{t}(t - 1) + I(t) - M_{t}(t) - U_{t}(t)
\]

\[X_{D}(t) = X_{D}(t - 1) + U_{D-1}(t) - M_{D}(t) - U_{D}(t) \text{ for } D = 2, \ldots, N
\]

and

\[X_{N+1}(t) = U_{N}(t)
\]

^2Stand basal area is approximated as \(\sum_{D=1}^{N} X_{D} Y_{2N}\).

One square foot of basal area per acre = 0.23 m²/ha.
where \( X_1 \) refers to the 6 in. (15.2 cm) class, \( X_2, \ldots, X_5 \) refer to the 8, 10, 12, \ldots\ in. (20.3, 25.4, 30.5, \ldots\ cm) classes, etc. and \((t-1)\) values are for the previous growth period. Cuts (assumed to be taken at the end of the growth period) can be introduced in the model simply by deducting the number of trees per acre harvested in each class from Eqs. [4]–[6]. Cuts influence stand growth by changing the distribution of trees at the end of a growth period and hence the magnitude of the growth components described by Eqs. [1]–[3] for the next growth period. From Eqs. [1]–[6] it is clear that the change in the number of trees in any given diameter class depends on the initial number of trees in all classes.

As an aid to yield analyses, rough cordwood volumes for each diameter class were expressed as:

\[
Y_{3d} = 0.000478 Y_{1d}^{1.92004} S^{0.335629} X_d - 0.03474 X_d
\]

\( r^2 = 0.97, \text{ Standard error } = 0.69 \)

Board foot volumes \( (Y_{4d}) \) were computed by applying board foot per cord conversion ratios \((BC_d)\) developed from Gevorkiantz (1950) to Eq. [7]:

\[
Y_{4d} = BC_d Y_{3d} \text{ for all } D.
\]

The model of Eqs. [1]–[8] can be used to simulate the growth and development of stands with or without harvesting. The structure of the model allows cutting only at intervals which are integer multiples of the growth period length. Using the model as a simulator, alternative harvesting schedules may be compared both on the basis of short-term yields and with respect to their longer term effects on stand structure. As an illustration, Table 1 gives the results of a 50 year simulation where harvesting was undertaken every 5 years so as to maintain cordwood volume production in the stand at 4% per annum. The harvesting was conducted by cutting in the largest classes first. The model might be employed at this stage to examine alternative stocking and stand structures by simulating a range of stocking–structure combinations such as would be considered in field experiments.

While this paper will deal exclusively with the growth model in its deterministic form, stochastic simulations are also possible. Such simulations could be based on the residual error variances obtained from fitting Eqs. [1]–[3]. Assuming an appropriate distribution for these errors, say, normal with zero means and variances as estimated, random disturbances could then be added to each growth component. A given harvesting simulation could then be rerun several times to obtain estimates of the variability of harvest yields and residual diameter distributions.

### Optimal Stand Structure and Stocking

#### A. Sustainable Distributions

On the assumption that sustained yield is the guiding objective of uneven-aged forest management, silviculturists have traditionally sought to indentify those types of uneven-aged stand structures in which . . . current growth can be removed periodically while maintaining the
diameter distribution and initial volume of the forest" (Meyer 1952). Based on observations of natural and managed stands which appeared to be in such an equilibrium state, Meyer (1952, 1953) and others have concluded that the 'optimal' diameter distribution for an uneven-aged stand follows de Liocourt’s rule. That is, that

\[ X_D(t) = k \times X_{D-1}(t) \text{ for } D = 2, \ldots, N \]

where \( k \) is some constant (less than one) which may vary by site and forest type. Such stands have been termed 'balanced' uneven-aged stands. Once \( k \) is determined and the maximum tree size set, the optimal distribution of trees for any given stocking level (e.g. volume or basal area level) can be determined using the \( N - 1 \) equations noted above, plus one additional equation giving the stocking level as the sum of volumes (or basal areas) in each diameter class.

In addition to these theoretically optimal diameter distributions, a great many distribution and stocking guides have been derived from field experimentation. In the Lake States, for example, guides developed by Eyre and Zillgitt (1950, 1953); (also Arbogast (1957)) from field experiments in northern Michigan have been widely applied to the northern hardwood type. Similar guides exist for certain southern pine and hardwood types and a number of other major forest types (see, for example, Putnam et al. 1960; and Reynolds 1969). These empirical guides often depart from de Liocourt-type distributions, particularly by including more trees in the smaller diameter classes. Unfortunately, the difficulties of field experimentation generally preclude an exhaustive examination of a broad range of both stocking and distribution alternatives. Consequently, some extrapolation of actual experimental results is required to develop general guides for optimal distributions for different stocking levels. With this limited approach or simulation, however, it is impossible to be certain that the 'best' combination of distribution and stocking was actually tested.

Whatever the particular form of the optimal diameter distribution, maintenance of a given distribution from one cutting cycle to the next clearly requires that just prior to cutting the number of trees in each diameter class be at least as large as the number at the start of the cycle. If it is assumed that the length of the cutting cycle is equal to the length of the growth period (5 years) in the model of Eqs. [4]–[6], then just prior to harvest we must have:

\[ X_D(t) \geq X_D(t-1) \text{ for all } D. \]

Any (initial) diameter distribution which satisfies this condition can be sustained indefinitely ( barring catastrophes) by harvesting the excess of \( X_D(t) \) over \( X_D(t-1) \). In the present study, distributions satisfying the above condition have been called 'sustainable'. An example of a sustainable distribution and the resulting (terminal) distribution at the end of the cutting cycle as projected by the growth model is shown in Fig. 1. Trees in the crosshatched region would be removed by cutting, while natural mortality and upgrowth stabilize the lower end of the distribution.

**B. Optimization Using the Growth Model**

A traditional problem in uneven-aged management has been the determination of the 'best' stocking of the stand at the start of the cutting cycle and the diameter distribution corresponding to this stocking level. What is 'best' will, of course, vary with management goals, but it is clear that the stocking and distribution decisions are interdependent. The growth model of Eqs. [1]–[8] affords a means of determining the optimal, sustainable diameter distribution for a given level of stocking, and by varying the stocking level, allows the development of the information needed to make the optimal stocking decision. To illustrate the methodology, the diameter distributions which maximize the value growth of the northern hardwood model for a range of initial basal areas were found. The problems considered may be written as:

\[
\max_{\{X_D(t)\}_{D=1}} \sum_{i=1}^{N+1} V_D \Delta_D
\]

subject to:

\[
\Delta_D \geq 0, D = 1, \ldots, N + 1
\]

'It is possible under a more complicated system of selection harvesting to have deficiencies in diameter classes at intermediate time points yet still maintain a given distribution over some longer time period. In this study attention is restricted to a system involving a single cut occurring at the end of each growth period.
Fig. 1. Example of sustainable diameter distribution. Difference between initial (solid line) and terminal (dashed line) distributions removed by harvest (cross-hatched area).

\[ \sum_{D} Y_{2D}X_{D}(t) = L \]  
\[ X_{D}(t) \geq 0, \ D = 1, \ldots, N \]

The objective function [9] is written with all measured diameter classes considered merchantable. The \( V_{i} \) are estimated stumpage values (in dollars) per tree as tabulated in the last column of Table 2. The \( \Delta_{D} \) are changes in the number of trees in the \( D^{th} \) diameter class over the cutting cycle. The first set of \( N + 1 \) constraints (Eq. [10]) are simply a restatement of the sustainability conditions for a single cut selection harvest treatment:

\[ \Delta_{D} = X_{D}(t + 1) - X_{D}(t) \geq 0 \text{ for all } D. \]

Constraint [11] requires that the sum of the basal areas in all classes at the start of the cutting cycle must equal some fixed level \( L \). (\( Y_{2n} \), as previously noted, is the average basal area per tree in class \( D \)). The stocking level (\( L \)) was varied between 60 and 120 ft\(^2\)/acre (13.8 and 27.6 m\(^2\)/ha) in successive solutions of the problem. Finally, constraints [12] disallow negative numbers of trees in the initial distribution.

Expressions [9] and [10] can be written as functions of the initial numbers of trees in each class (\( X_{D}(t) \)) if appropriate substitutions are

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*Tree values were derived from Wisconsin pulpwood stumpage and sawlog prices as of summer 1972 and assumptions on the distribution of trees by butt log grade in the stand and log grade yields by butt log grade and d.b.h. The values are approximations intended to serve only as an illustrative example.*
### Table 2. Optimal diameter distributions, growth and growing stock data from value growth maximization subject to basal area constraint

<table>
<thead>
<tr>
<th>Number of trees/acre subject to stocking level constraint (basal area/acre)</th>
<th>Lake States northern hardwoods guide</th>
<th>Cords/tree</th>
<th>Board feet/tree</th>
<th>Value/tree</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>60</td>
<td>70</td>
<td>80</td>
<td>90</td>
</tr>
<tr>
<td>Diameter class (in)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>159.1</td>
<td>119.4</td>
<td>103.6</td>
<td>96.7</td>
</tr>
<tr>
<td>8</td>
<td>25.3</td>
<td>36.6</td>
<td>43.1</td>
<td>50.0</td>
</tr>
<tr>
<td>10</td>
<td>12.3</td>
<td>21.0</td>
<td>27.6</td>
<td>33.1</td>
</tr>
<tr>
<td>12</td>
<td>10.0</td>
<td>16.8</td>
<td>22.1</td>
<td>26.4</td>
</tr>
<tr>
<td>14</td>
<td>2.9</td>
<td>5.0</td>
<td>6.7</td>
<td>8.1</td>
</tr>
<tr>
<td>16</td>
<td>1.0</td>
<td>1.7</td>
<td>2.3</td>
<td>2.8</td>
</tr>
<tr>
<td>18</td>
<td>0.5</td>
<td>0.8</td>
<td>1.1</td>
<td>1.3</td>
</tr>
<tr>
<td>20+</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>Five year value growth ($)</td>
<td>35.00</td>
<td>37.78</td>
<td>40.39</td>
<td>42.76</td>
</tr>
<tr>
<td>Five year board feet growth</td>
<td>628</td>
<td>631</td>
<td>1049</td>
<td>1154</td>
</tr>
<tr>
<td>Five year cord growth</td>
<td>3.90</td>
<td>3.71</td>
<td>3.73</td>
<td>3.80</td>
</tr>
<tr>
<td>Value of stocks ($)</td>
<td>72.98</td>
<td>114.10</td>
<td>146.20</td>
<td>173.38</td>
</tr>
<tr>
<td>Board feet stocking</td>
<td>1518</td>
<td>2540</td>
<td>3378</td>
<td>4053</td>
</tr>
<tr>
<td>Cord stocking</td>
<td>12.22</td>
<td>15.64</td>
<td>18.63</td>
<td>21.37</td>
</tr>
<tr>
<td>Dual of basal area constraint</td>
<td>+0.283</td>
<td>+0.271</td>
<td>+0.249</td>
<td>+0.224</td>
</tr>
<tr>
<td>Average annual value growth (%)</td>
<td>8.2</td>
<td>5.9</td>
<td>5.9</td>
<td>4.5</td>
</tr>
<tr>
<td>Marginal value growth</td>
<td>2.78</td>
<td>2.61</td>
<td>2.37</td>
<td>2.12</td>
</tr>
<tr>
<td>Marginal value of growing stock</td>
<td>41.18</td>
<td>32.10</td>
<td>27.18</td>
<td>27.24</td>
</tr>
<tr>
<td>Marginal value growth percentage</td>
<td>1.3</td>
<td>1.6</td>
<td>1.7</td>
<td>1.5</td>
</tr>
</tbody>
</table>

*Metric equivalents are: 1 in. = 2.54 cm; 1 tree/acre = 2.47 trees/ha; 1 cord/acre = 8.96 m³/ha (overall measure of stacked roundwood); 1 ft² of basal area/acre = 0.23 m² basal area/ha.

*Derived from Arbogast (1957), data not shown for 20 in. and larger classes. Basal area is 8.31 ft².

*The value of the "dual" of the basal area constraint may be interpreted as the prospective change in the value of the objective function for a one unit increase in the level of the constraint. For example, at 60 ft² basal area, an increase to 61 ft² would lead to approximately a $0.283/acre increase in the value growth. Note that a 10 ft² increase to 70 ft² actually increased value growth by $2.78 while the estimated increase from the dual was $2.83. The difference arises due to the curvilinearity of the objective function.
made using Eqs. [1]–[6]. The resulting expressions are nonlinear in the $X_{n}(t)$. The problem posed then is to maximize an objective function composed of the sum of $N + 1$ nonlinear terms subject to a set of linear and nonlinear constraints. The solution procedure used was a modification of the gradient projection method. Optimization results for site $S = 60$ are shown in Table 2.

Relative to existing guides for the management of northern hardwoods (Arbogast 1957) the distributions in Table 2 all have many more trees in the smaller diameter classes and fewer trees in the larger classes. Average annual board foot volume yields vary from 125 to 320 bd. ft/acre (48 to 123 m$^3$/ha) over the basal area levels considered. In the table, initial stocking higher than 110 ft$^2$ (25.3 m$^2$/ha) yields lower board foot volume growth. Both value and board foot volume growth actually peak at about 115 ft$^2$ (26.4 m$^2$/ha). As expected (see Jacobs 1968) cubic volume growth, as estimated in cords, remains relatively stable over the range of stocking levels examined. At stocking levels from 60 to 110 ft$^2$ (13.8 to 25.3 m$^2$/ha), timber harvests (the difference between initial and terminal distributions) yield both cordwood and sawlog volumes. At 120 ft$^2$ (27.6 m$^2$/ha), only sawlog harvest is realized.

Using Table 2 and the approach described by Duerr and Bond (1952) the optimal economic stocking level is that at which marginal value growth percentage equals the manager's alternative rate of return. The marginal value growth percentages for the illustration are shown in the lower portion of the table. They increase to a peak and decline, since the value growth – value of stocking relation has an inflection at about 90 ft$^2$ (20.7 m$^2$/ha) of basal area. For this example, the marginal value growth percentages are all rather low.

C. Modifications and Extensions

Use of the growth model for defining optimal stand structure need not be restricted to the specific problem examined above. Virtually any objective function (with continuous first derivatives) definable in terms of the initial numbers of trees in each diameter class could be employed. Initial stocking constraint(s) can be defined in terms of volume or numbers of trees per acre either as equalities or inequalities. They might also be eliminated entirely if the unconstrained objective maximizing solution is sought. It is important to note, however, that so long as the sustaintability and nonnegativity constraints are retained some problems posed may not be feasible. Some experimentation is usually required to determine the feasible ranges for different types of constraints.

In the present study, objective functions which maximize growth of cordwood, board foot volume, and several alternative specifications of tree value were examined. For given initial stocking and sustainability constraints the resulting optimal distributions varied markedly. The sensitivity of results to the form of the tree value function in the preceding example is of particular interest. Figure 2 compares optimal distributions for stands of 80 and 120 ft$^2$ (18.4 and 27.6 m$^2$/ha) initial basal area stocking using the value function from the problem described above and an alternative value function which gives substantially more weight to larger diameter classes. The effects on the optimal distributions are clear from the figure and correspond to anticipated differences between stands managed on a sawlog objective and those managed for both cordwood and sawlogs. The shifts in the distributions as stocking increases are also characteristic. For the mixed cordwood–sawlog objective the distribution rotates counterclockwise as stocking increases, while with the sawlog objective the distribution shifts vertically with no overlap in the range of diameters examined. The Lake States northern hardwood management guide (Arbogast 1957) distribution is shown in the figure for comparison. The growth period and cutting cycle in the present study were both assumed equal at 5 years. Examination of longer cutting cycles could be carried out by: (i) reestimation of the growth model (Eqs. [1]–[3]) for the appropriate cycle length, or (ii) if the desired cycle is an integer multiple of 5 years, sequential solution of the growth Eqs. [1]–[6]. The first
approach would require the availability of growth data for a time period equal to the length of the desired cutting cycle. The latter method leads to increasingly complicated growth equations (for the $\Delta n$) as the number of successive growth periods increases.

A third and somewhat more tractable approach for cutting cycles which are integer multiples of 5 years would be as follows. Let $G(X(t))$ be the diameter distribution resulting from the substitution of some initial distribution, described by the vector $X(t)$, into the growth model of Eqs. [1]–[6]. For simplicity examine a 10 year cutting cycle. The problem is represented diagrammatically on Fig. 3. Some initial distribution, $X(0)$, grows to $X(1) = G(X(0))$ which in turn grows to $X(2) = G(X(1))$ by the end of the 10 year cycle. The objective, as before, of maximizing value yield (given by the function $V(X(t))$) over the cycle is:

max $\quad V(X(2) - X(0)) = V(G(X(1)) - X(0))$

subject to the constraints

$X(1) = G(X(0))$ \hspace{1cm} $X(1)$ results from $X(0)$,

$G(X(1)) \geq X(0)$ \hspace{1cm} sustainability,

$\sum_{p} Y_{2p} X_{p}(0) = L$ \hspace{1cm} initial basal area constraint,

and

$X(0), X(1) \geq 0.$

Fig. 3. Representation of optimal distribution when cutting cycle is longer than one growth period. $X(0)$ is to be found so as to maximize cut.
In this problem the desired initial distribution \( X(0) \) is determined and also a superfluous intermediate distribution \( X(1) \). Twice as many variables must be determined here and there are \( N \) more constraints than in the problem of Eqs. [9]–[12]. Much effort is saved, however, relative to the successive solution approach and the same optimization procedures can be applied as were used in the original 5-year problem. As noted, however, cutting cycles are limited to those which are integer multiples of the growth period.

In the optimization problems described above, it was assumed that only trees in the 18 in. (45.7 cm) and smaller classes were retained in the residual stand. For a 5 year cycle this allows trees up to the 20 in. (50.8 cm) class in the precut distributions (maximum diameter of 20.9 in. (53.1 cm)). The existence of larger trees can be accommodated by appropriately increasing \( N \) in Eqs. [1]–[6] and augmenting the objective function and constraints in the optimization problem. For the present study inclusion of the 18 in. (45.7 cm) class approaches the limits of the growth data used in estimating the model. Trees larger than 22.7 in. (57.7 cm) were not observed in the original sample and trees in the 20 in. (50.8 cm) class were extremely rare.\(^7\)

While not considered in detail here, the general analytical approach used might also allow incorporation of aesthetic, wildlife, and socio-economic factors in the formulation of management guides. These factors might be entered as constraints (e.g. lower bounds on the numbers of ‘large’ trees per acre for aesthetic purposes) or as the objective function if quantifiable. Finally, individual species were lumped together in the growth model of Eqs. [1]–[8]. If data are available, separate treatment might be given species groups by simply expanding the growth model and the number of variables included in the optimization problem.

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\(^7\) Including a sufficiently large number of diameter classes in the original problem formulation allows determination of the maximum desirable diameter for trees in the stand. For the problems in Table 2, allowing trees of up to 22 in. (55.9 cm) diameter in the initial distribution led to changes of less than 1 tree/acre (2.5 trees/ha) in the 6–18 in. (15.2–45.7 cm) classes while less than 0.5 trees/acre (1.25 trees/ha) were added in the 20 and 22 in. (50.8 and 55.9 cm) classes.

**Optimal Conversion to Sustained Yield**

**A. The Transition Problem**

The approach discussed in the preceding sections considers the development of optimal sustainable initial diameter distributions and equilibrium diameter class cutting rules for given forest type, site, cutting cycle, and economic constraints and conditions. As a practical matter, however, many existing stands will differ in structure from desired equilibrium conditions of stocking and diameter distribution. Some methodology is needed for determining the appropriate cutting schedule during the transition period in which the stand is brought from its actual, irregular condition to a desired sustainable condition. As was true of sustainable distributions, there are in general many possible harvesting schedules which will effect the desired transition from given initial to desired terminal stand conditions depending upon the length of the transition period and management objectives. The concern is to find the ‘best’ transitional cutting schedule for a particular set of circumstances.

A similar problem exists in the transitional management of even-aged stands. Nautiyal and Pearse (1967) have examined the even-aged conversion problem in a linear programming context. They noted the interrelation of the choice of terminal conditions (rotation age in the even-aged case) and length of the conversion period and their impact on an objective of maximizing the present worth of returns during the transition period and all subsequent periods. Using data presented by Chappell and Nelson (1964), Amidon and Akin (1968) have shown the applicability of dynamic programming in determining optimal stocking and rotation as well as conversion cutting schedules in individual even-aged loblolly pine stands. In these and the present uneven-aged case, the problem may be usefully envisioned as one of determining an optimal trajectory between some given initial stand conditions and some fixed terminal conditions (or simply a point in time if no terminal conditions are specified). If the ‘state’ of an uneven-aged stand at some initial point is given by a vector of number of trees per acre by diameter class, \( X(0) = [X_1(0), X_2(0), \ldots, X_N(0)] \), then the fixed end point problem involves converting this vector to some desired vector, \( X^*(T) = [x_1^*(T), \ldots, x_N^*(T)] \).
$X^*(T)$, over a conversion period of length $T$. The path taken between $X(0)$ and $X^*(T)$ may be called a trajectory.

Given the initial diameter distribution, $X(0)$, several actions are possible during the first and each subsequent growth interval of the conversion period. The stand may be allowed to grow with no harvesting or any of a number of cutting schemes, denoted by a vector $C(t)$, and might be applied to the several diameter classes. What is sought is a set of diameter distributions, $X(0), X(1), \ldots, X(t), \ldots, X(T)$, and a set of cutting rules, $C(0), C(1), \ldots, C(t), \ldots, C(T)$, which describe the structure of the stand and the ‘controls’ or cuts applied in each growth period to convert $X(0)$ to $X^*(T)$ in some optimal fashion.

Mathematically the problem may be posed as:

$$\max \sum_{t=1}^{T-1} V_t(X(t), C(t)) + V_T(C(T), X^*(T))$$

subject to

$$13 \quad X(t) = X(t-1) + \Delta(t) - C(t),$$

$$14 \quad X(0), \text{ given initial stand structure,}$$

$$15 \quad X(T) = X^*(T), \text{ given desired terminal stand structure,}$$

$$16 \quad X(t) \geq 0, \text{ for all } t,$$

$$17 \quad C(t) \geq 0, \text{ for all } t.$$

The objective function [13] might be expressed, for example, as the present value of all management activities during the conversion period plus the returns from an infinite series of future equilibrium cutting cycles. The future cutting cycles would begin $T$ periods hence, after the stand is converted to some sustainable or balanced condition ($X^*(T)$). Appropriate value functions, $V_t$, would involve both cut and the current stand structure since the $C(t)$ yield revenues while costs might, in general, depend on the level of growing stock as determined from $X(t)$. Equation [14] is a vector-valued function which says simply that the number of trees in each class at the end of period $t$, $X(t)$, equals the numbers at the start of the period, $X(t-1)$, plus net natural growth, $\Delta(t)$, less removals, $C(t)$. Conditions [17] and [18] disallow negative numbers of trees in any diameter class or cut operation, respectively. Using the growth model of Eqs. [1]–[6], the $\Delta(t)$ of Eq. [12] can be expressed as a function of $X(t-1)$.

At least three (analytic) approaches are available to solve problems such as this. The problem as posed in expressions [13]–[18] is a statement of a fairly general discrete time variational problem, and solution by the methods of optimal control theory is at least theoretically possible (see Dreyfus (1972) for a simplified discussion of this approach). Such an approach would allow the $X(t)$ and $C(t)$ to be treated as continuous variables (i.e., discretization of the state and control variables would not be necessary) but unfortunately the computational process becomes extremely burdensome since both $C(t)$ and $X(t)$ are constrained to be nonnegative.

Alternatively an approach analogous to that used by Amidon and Akin (1968) might be attempted. In this case it would be necessary to define an $(N+1)$ dimensional network through which a path must be found from $X(0)$ to $X^*(T)$ in some optimal fashion. The computations necessary are simple relative to the control theoretic approach but the number of possible paths which must be considered at any given stage in the conversion period is enormous.

A third approach, motivated by the preceding discussion of longer cutting cycles, appears to be computationally feasible and has considerable intuitive appeal. The method can be best illustrated by a simplified problem involving only two growth intervals over the conversion period. (See Fig. 4). Beginning with a distribution $X(0)$, a conversion cutting schedule $(C(0), C(1), C(2))$ is sought which leads to $X^*(2)$ so as to maximize some objective. Assuming, for simplicity, that only the levels of cut $(C(t))$ influence costs and revenues during the transition, the objective can be written as:

\[ \text{Fig. 4. Representation of optimal conversion cutting problem for a two period (10 year) case. Initial distribution is } X(0) \text{ and the terminal distribution is } X^*(2). \]
### Table 3. Illustrative example of optimal conversion cutting schedule with an objective of maximizing present worth of conversion period returns. Conversion period is 10 years in length with cutting allowed at years 0, 5, and 10 to reach desired terminal distribution in column (9).

<table>
<thead>
<tr>
<th>Diameter (in.)</th>
<th>(1) Initial (starting) distribution (trees/acre)</th>
<th>(2) Cut at year 0 (trees/acre)</th>
<th>(3) Residual at year 0 (trees/acre)</th>
<th>(4) Initial distribution at year 5 (trees/acre)</th>
<th>(5) Cut at year 5 (trees/acre)</th>
<th>(6) Residual at year 5 (trees/acre)</th>
<th>(7) Initial distribution at year 10 (trees/acre)</th>
<th>(8) Cut at year 10 (trees/acre)</th>
<th>(9) Terminal (desired) distribution (trees/acre)</th>
<th>(10) Total present value of conversion cuts ($/acre)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>140.00</td>
<td>0.00</td>
<td>140.00</td>
<td>138.46</td>
<td>0.00</td>
<td>138.46</td>
<td>135.41</td>
<td>31.80</td>
<td>103.61</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>50.00</td>
<td>0.00</td>
<td>50.00</td>
<td>63.69</td>
<td>0.00</td>
<td>37.87</td>
<td>25.82</td>
<td>46.49</td>
<td>43.09</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>28.00</td>
<td>0.00</td>
<td>28.00</td>
<td>31.37</td>
<td>0.00</td>
<td>31.37</td>
<td>27.60</td>
<td>0.00</td>
<td>27.60</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>23.00</td>
<td>3.81</td>
<td>19.19</td>
<td>20.37</td>
<td>0.00</td>
<td>20.37</td>
<td>22.11</td>
<td>0.00</td>
<td>22.11</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>7.00</td>
<td>3.09</td>
<td>3.91</td>
<td>8.97</td>
<td>5.87</td>
<td>3.10</td>
<td>9.01</td>
<td>2.44</td>
<td>6.67</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>6.00</td>
<td>6.00</td>
<td>0.00</td>
<td>1.90</td>
<td>0.01</td>
<td>1.89</td>
<td>2.29</td>
<td>0.00</td>
<td>2.29</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>2.00</td>
<td>2.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>1.07</td>
<td>0.00</td>
<td>1.07</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Present value of conversion cutting ($/acre)</td>
<td>60.81</td>
<td>31.76</td>
<td>31.76</td>
<td>99.23</td>
<td>6.66</td>
<td>99.23</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Present value from immediate conversion ($/acre)</td>
<td>32.94</td>
<td>31.64</td>
<td>24.79</td>
<td>89.37</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

*Metric equivalents are: 1 in. = 2.54 cm; 1 tree/acre = 2.47 trees/ha.

**Note:**
- Present value of cutting during conversion = $99.23$/acre.
- Present value of all subsequent cuts = ($540.59(1.05^{10} - 1)$) / (1.05) = $89.75$/acre.
- Total present value of management program = $189.98$/acre.
\[
\begin{align*}
\max & V_0(C(0)) + V_1(C(1)) + V_2(C(2)) \\
& \quad + V_\infty(X^*(2)) \\
\text{or} & \max_{(X_\delta(0), X_\iota(1))} V_0(X(0) - X_\delta(0)) + V_1(X_\iota(1)) \\
& \quad - X(1) + V_2(X_\rho(2) - X^*(2)) + V_\infty(X^*(2))
\end{align*}
\]

where the \( V_i \) are value operators, \( e.g. \) present worths, and \( V_\infty(X^*(2)) \) denotes the value of terminal stocks or of the ensuing management practices, \( e.g. \) the present value of an infinite series of equilibrium harvests. The variables with respect to which maximization takes place are only the distributions \( X_\delta(0) \) and \( X_\iota(1) \), since \( X_\rho(1) = G(X_\delta(0)) \) and \( X_\rho(2) = G(X_\iota(1)) \) using the notation of section "C. Modifications and Extensions". The constraints which must be satisfied are that (i) negative cutting is not allowed, \( i.e. \) \( C(t) \geq 0 \) for all \( t \), and (ii) all \( X(t) \geq 0 \).

The complete problem can be stated as:

\[\text{[19]} \max_{(X_\delta(0), X_\iota(1))} V_0(X(0) - X_\delta(0)) \]
\[+ V_1(G(X_\delta(0)) - X(1)) + V_2(G(X(1))) \]
\[- X(1) + V_\infty(X^*(2)) \]

subject to

\[\text{[20]} C(0) = X(0) - X_\delta(0) \geq 0, \]
\[\text{[21]} C(1) = X_\rho(1) - X(1) = G(X_\delta(0)) \]
\[- X(1) \geq 0, \]
\[\text{[22]} C(2) = X_\rho(2) - X^*(2) = G(X(1)) \]
\[- X^*(2) \geq 0, \]

and

\[\text{[23]} X_\delta(0), X(1) \geq 0. \]

With \( X(0) \) and \( X^*(2) \) fixed, this is nothing more than an expansion of the type of problem solved in the first section of this paper. The length of the interval from say \( t = 0 \) to \( t = 1 \) is fixed at the length of the growth period of the biological model (5 years in the present case), and the 'starting' distributions in each period \( (X_\delta(0) \) and \( X(1)) \) are the variables to be determined.\(^8\)

The problem of expressions [19]–[23] is analogous to the usual linear programming (L.P.) formulation for optimizing the conversion of an even-aged forest (\( c.f. \) Nautiyal and Pears 1967). The objectives are identical. In the L.P., even-aged case the activities are generally taken as acres harvested by age class and time period, rather than residual acres by age class and time period. The latter form is more comparable to the uneven-aged formulation, but it is clear that the two formulations are interchangeable. In the L.P., even-aged case there are constraints which (i) restrict harvesting in any given age class and time period to be no larger than the acreage available, and (ii) require that the forest be fully regulated by the end of the conversion period.

In the uneven-aged problem constraints \([20]–[23]\) are analogous to the former, while the latter is implicit in the problem formulation \((i.e. \) the cut taken at time 2 is that necessary to obtain the desired \( X^*(2) \)).

\section*{B. Example of Conversion Schedule Determinations}

Table 3 presents the results of a specific conversion problem to illustrate the approach suggested in Eqs. [19]–[23]. The initial distribution \( (X(0)) \) is given in column (1) and the desired terminal distribution \( (X^*(2)) \) in column (9) of Table 3. The objective was to maximize the present value of yields during the conversion period plus the (constant) sum of all future equilibrium yields once the stand is fully converted 10 years hence. A 5% discount rate was assumed, and the per tree values shown in Table 2 were appropriately adjusted to give present values per tree. The terminal distribution corresponds to the 80 ft\(^2\) (18.4 m\(^2\)/ha) basal area stock example in Table 2.

The indicated optimal cutting schedule by diameter class is given in columns (2), (5), and (8). The present worth of material cut during the 10 year conversion period totalled $99.23/acre ($245.10/ha). Adding the value of subsequent equilibrium cuttings at five year intervals, the total worth of conversion and sustained yield management would be $188.98/acre ($466.78/ha). It will be noted that while the stand could have been converted immediately to the desired terminal distribution, it is uneconomic to do so. As indicated at the bottom of Table 3, immediate conversion increases the present value of the last cutting in

\(^{8}\text{Appropriate modifications in this problem would allow direct treatment of the more complicated selection cutting schemes noted in footnote 3.}\)
the conversion period but sharply reduces the value of the initial cut. The result is a net reduction of $9.86/acre ($24.35/ha) in the present worth of conversion period revenues.

C. The Characteristics of Conversion Cutting Schedules

The specific form of the optimal conversion cutting schedule resulting from solution of the problem presented in section “A. Transition Problem” will vary with the objective function used, length of the conversion period, initial and desired terminal stand conditions. Some general features of conversion cutting schedules can, however, be deduced for certain economic objectives. If it is specified that the objective is to maximize the present worth of all transitional and subsequent equilibrium yields, then it seems reasonable to anticipate a relationship between length of transition period \( T \) and terminal stocking level \( S \) analogous to the conversion period – rotation results of Nautiyal and Pearse (1967). In such a problem the planning horizon is infinite and aggregate returns are composed of two elements:

\[
C(S, T) = \text{the present worth of returns from cutting during the conversion period to reach a terminal stocking level} \ (S) \ \text{and a given diameter distribution in} \ T \ \text{5-year growth periods, and}
\]

\[
E^*(S, T) = E(S)(1 + i)^T \ \text{the present worth of an infinite series of equilibrium harvests} \ (E(S)) \ \text{from a sustainable distribution of initial density} \ S \ \text{beginning} \ T \ \text{growth periods hence (or} \ T \ \text{years).}
\]

Aggregate returns, then, are:

\[
A(S, T) = C(S, T) + E^*(S, T).
\]

It is clear that the slope of \( A(S, T) \) as either \( S \) or \( T \) is varied will depend on the signs of the partial derivatives of \( C \) and \( E^* \). In the case of varying desired stand density at the end of the conversion period \( E^* \) is likely to have a peak with respect to \( S \), so that \( \partial E^*/\partial S = E_{S}^* \) will be positive and then negative as \( S \) increases over some range. \( C \) on the other hand should decline steadily as \( S \) increases (i.e. \( C_{S}^* < 0 \)), since more (and larger) trees are required in the sustainable distribution at the end of the conversion period and hence fewer large and more valuable trees can be cut during the conversion. \( A(S, T) \) may then yield a peaked function over \( S \) if \( C_S > -E_{S}^* \) in some range of \( S \), a result similar to that proposed by Nautiyal and Pearse for the even-aged conversion period as rotation age is varied and conversion period held constant.

For a fixed \( S \), increasing \( T \) clearly leads to a monotonic decline in \( E^* \), i.e., \( E_{S}^* < 0 \) for all \( T \). \( C(S, T) \) should, however, steadily increase with \( T \). As the conversion period is lengthened, larger cuts are possible early in the period with concomitant increases in \( A(S, T) \). In general actual conversion cutting (or lack of cutting) should occur as close as possible to the end of the period given the larger discounting factors as time proceeds. Under such conditions \( A(S, T) \) will have a positive slope so long as \( C_T > -E_{S}^* \). \( A(S, T) \) may approach some asymptotic upper limit set by a strategy of never converting the forest to sustained yield.


